

# Umbral Calculus and Cancellative Semigroup Algebras\*

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*Dedicated to the memory of Gian-Carlo Rota*

## Abstract

We describe some connections between three different fields: combinatorics (umbral calculus), functional analysis (linear functionals and operators) and harmonic analysis (convolutions on group-like structures). Systematic usage of cancellative semigroup, their convolution algebras, and *tokens* between them provides a common language for description of objects from these three fields.

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It is a common occurrence in combinatorics for a simple idea to have vast ramifications; the trick is to realize that many seemingly disparate phenomena fit under the same umbrella.

Richard P. Stanley, [21].

## 1 Introduction

We discuss some interactions between representation theory of semigroups, umbral calculus in combinatorics, and linear operators in functional analysis. Our main bridge between them is the notion of cancellative semigroups, convolution algebras over them and tokens from such a semigroup to another.

We look for a vocabulary, which is reasonably general and allows us to translate without alterations as much about umbral calculus as possible. The language we select relies on convolution algebras. Such algebras arise from essentially different sources like groups, posets, or linear operators. The goal of the present paper is to study properties of convolution algebras, which are independent from their origins.

The *umbral calculus* was known long ago but it was not well understood till very recent time. During its history it took many different faces: linear functionals [14], Hopf algebras [13], axiomatic definition [19] to list only few different interpretations.

The goal of our paper is to give a model, which realizes described features by means of common objects. What is a purpose of such models? One can think on the Poincaré model of non-Euclidean geometry in a disk. The model does not only demonstrate the logical consistency of Lobachevski's geometry. Another (no less important) psychological function is to express the unusual geometry via objects of the intuitive classic geometry. The logical consistence of umbral calculus is evident (from papers [13, 14, 19] for example). But is our feeling that some more links with fundamental objects are still desirable.

The paper presents basic definitions and properties of cancellative semigroups, tokens, convolution algebras, and their relations with umbral calculus. We will illustrate our consideration mostly by the simplest example: the cancellative semigroup of non-negative integers  $\mathbb{N}_+$ . More non-trivial examples will be present elsewhere.

The layout is as follows. In the next Section we sketch three different interpretations of umbral calculus from papers [13, 14, 19]. Section 3 introduces cancellative semigroups, convolutions, and tokens and their basic properties. We use them in Section 4 to describe principal combinatorial objects like delta families, generating functions, and recurrence operators. We conclude the paper with Section 5 which links our construction with three realizations of umbral calculus from [13, 14, 19] recalled in Section 2.

## 2 The Umbral Calculus

The umbral calculus was put on the solid ground by G.-C. ROTA and collaborators [13, 14, 11, 19] in three different ways at least. We repeat here some essential definitions from these papers.

### 2.1 Finite Operator Description

The three ways is based essentially on two fundamental notions [16]:

1. A *polynomial sequence of binomial type*, that is, a sequence of polynomials  $p_n(x)$  ( $\deg p_n = n$ ) with complex coefficients, satisfying the identities:

$$p_n(x+y) = \sum_{k=0}^n p_k(x)p_{n-k}(y). \quad (2.1)$$

Our definition is different from the original one—we take polynomial divided by  $n!$ . Our choice will be explained shortly in connection with the group property. Such form of the defining identity together with a more accurate name *polynomial sequence of integral type* were used in [4]. Paper [10] also start from this formula, the polynomials are called there *convolution polynomials* and their investigation is made with a help of *Mathematica* software.

2. A *shift invariant operator*, namely a linear operator  $S$  on the vector space  $\mathcal{P}$  of all such polynomials, which commutes with the ordinary derivative  $Dp(x) = p'(x)$ , that is, an operator  $S$  with the property that  $TDp(x) = DTp(x)$  for all polynomials  $p(x)$ . Within shift invariant

operators the following one plays an exceptional role: an operator  $Q$  is said to be a *delta operator* (associated to a polynomial sequence  $p_n(x)$ ) when

$$Qp_n(x) = p_{n-1}(x) \text{ or more general } Q^k p_n(x) = p_{n-k}(x). \quad (2.2)$$

## 2.2 Hopf Algebras Description

There is a canonical isomorphism between shift invariant operators and linear functionals (see Proposition 3.11), thus we can think on powers of delta  $Q^k$  as functionals  $q_k$  on polynomials connected with associated polynomial sequence  $p_n$  by the *duality*:

$$\langle q_k, p_n(x) \rangle = \delta_{kn}, \quad (2.3)$$

which easily follows from (2.2). Here  $\delta_{kn}$  is the Kronecker delta<sup>1</sup>. We are going to show that delta operators and polynomial sequences is not only the dual notions, but two faces of the same object—tokens between cancellative semigroups (Examples 3.15 and 3.16).

## 2.3 The Semantic Description

This subsection contains definitions from [18, 19].

**Definition 2.1** [18] An *umbral calculus* is an ordered quadruple, consisting of

1. A commutative integral domain  $D$  whose quotient field is of characteristic zero.
2. An alphabet  $A$  whose elements are termed *umbrae* and denoted by Greek letters. This alphabet generates the polynomial ring  $D[A]$ , elements of which are termed *umbral polynomials*.
3. A linear functional  $\text{eval} : D[A] \rightarrow D$ , such that  $\text{eval}(1) = 1$ , i.e.,  $\text{eval}$  leaves  $D$  fixed, and such that

$$\text{eval}(\alpha^i \beta^j \cdots \gamma^k) = \text{eval}(\alpha^i) \text{eval}(\beta^j) \cdots \text{eval}(\gamma^k) \quad (2.4)$$

where  $\alpha, \beta, \dots, \gamma$  are distinct umbrae

4. A distinguished umbrae  $\epsilon$  in  $A$ , such that  $\text{eval}(\epsilon^i) = \delta_{i,0}$ .  $\epsilon$  is sometimes called the augmentation.

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<sup>1</sup>We use the same letter  $\delta$  to denote also the Dirac delta function. The Kronecker delta  $\delta_{kn}$  and the Dirac function  $\delta(x)$  can be distinguished by their arguments.

**Definition 2.2** [18] Let  $f$  and  $g$  be two umbral polynomials in  $D[A]$ . Then  $f$  and  $g$  are said to be *umbrally equivalent*, written  $f \simeq g$  when  $\text{eval}(f) = \text{eval}(g)$ .

**Definition 2.3** [18] Two umbral polynomials  $p$  and  $q$  are termed *exchangeable*, written  $p \equiv q$ , when  $p^n \simeq q^n$  for all  $n \geq 0$ .

### 3 Cancellative Semigroups and Tokens

#### 3.1 Cancellative Semigroups: Definition and Examples

Our consideration is based on the following notion of *semigroups with cancellation* or *cancellative semigroups* [3, § IV.1.1]:

**Definition 3.1** A semigroup  $C$  is called a *left (right) cancellative semigroup* if for any  $a, b, c \in C$  the identity  $ca = cb$  ( $ac = bc$ ) implies  $a = b$ . A *cancellative semigroup* is both left and right cancellative semigroup.

Equivalently a left (right) cancellative semigroup is defined by the condition that for arbitrary  $a, b \in C$  the equation  $a \cdot x = b$  ( $x \cdot a = b$ ) has at most one solution (if any). We hope that readability of the paper will be better if we use “*c-semigroup*” to denote “cancellative semigroup”. We also use the notion of *c-set*  $C$  (which is weaker than c-semigroup) by allowing the multiplication  $(a, b) \mapsto ab$  on  $C$  be defined only on a proper subset of  $C \times C$ .

**Definition 3.2** An element  $e$  is called a (left, right) *source* of a (left, right) c-semigroup  $C$  if both equations (the first equation, the second equations)  $x \cdot e = b$ ,  $e \cdot x = b$  do (does, does) have a (unique) solution for any  $b \in C$ .

Obviously, a (left, right) identity on  $C$  will be a (left, right) source.

We still denote the unique-if-exist solutions to equations  $x \cdot a = b$  and  $a \cdot x = b$  by  $[ba^{-1}]$  and  $[a^{-1}b]$  correspondingly. Here the braces stress that both  $[ba^{-1}]$  and  $[a^{-1}b]$  are *monosymbols* and just “ $a^{-1}$ ” is not defined in general. Let us also assume that all c-semigroups under consideration can be equipped by an *invariant measure*  $db$ , namely  $db = d(ab)$  for all  $a \in C$ . We will not discuss herein conditions for its existence or possible modifications of our constructions for the case of *quasi*-invariant measures.

C-semigroups were investigated as an algebraic object in connection with groups. It is particularly known that any commutative c-semigroup can be

embedded in a group but this is not necessary true for a non-commutative c-semigroup [3, § IV.1.1]. Our motivation in this object is the following. If we have a group  $G$  then the important associated object is the left regular (linear) representation by “shifts”  $\pi_g f(h) = f(hg)$  in  $\mathcal{L}_p(G, d\mu)$ . Then one can introduce their linear span—convolutions:

$$Kf(h) = \int_G k(g)\pi_g dg f(h) = \int_G k(g)f(hg) dg. \quad (3.1)$$

Thus one may consider the composition of convolutions  $K_2 K_1$  with two kernels  $k_1(g), k_2(g) \in \mathcal{L}_1(G)$ :

$$\begin{aligned} K_2 K_1 f(h) &= \int_G k_2(g_2) \int_G k_1(g_1) f(hg_2g_1) dg_1 dg_2 \\ &= \int_G \left( \int_G k_2(g_2) k_1(g_2^{-1}g) dg_2 \right) f(hg) dg \end{aligned}$$

where  $g = g_2g_1$ . Thereafter it again is a convolution with the kernel

$$k(g) = \int_G k_2(g_2) k_1(g_2^{-1}g) dg_2. \quad (3.2)$$

An important role in the consideration is played by the unique solvability of equation  $g_1g_2 = g$  with respect to  $g_1$ . For an algebraic structure weaker than a group, if the inverse  $g^{-1}$  is not defined one can try to define a composition of convolutions by the formula:

$$(k_1 * k_2)(h) = \int \int_{g_1g_2=h} k_1(g_1) k_2(g_2) dg_1 dg_2. \quad (3.3)$$

However such a definition generates problems with an understanding of the set double integral (3.3) is taken over and the corresponding measure. These difficulties disappear for left c-semigroups: for given  $b$  and  $a$  there is at most one such  $x = [a^{-1}b]$  that  $ax = b$ . Thus we can preserve for c-semigroup  $C$  definition (3.2) with a small modification:

$$(k_2 * k_1)(b) = \int_C k_2(a) k_1([a^{-1}b]) da \quad (3.4)$$

To avoid problems with the definition of convolution we *make an agreement* that for any function  $f(c)$

$$f([a^{-1}b]) = 0 \quad \text{when } [a^{-1}b] \text{ does not exist in } C \quad (3.5)$$

(i.e., the equation  $ax = b$  does not have a solution). Our agreement is equivalent to introduction of incidence algebra of functions  $f(a, b)$ , such that  $f(a, b) = 0$  for  $a > b$  for a poset (see Example 3.7).

**REMARK 3.3** As soon as one passes directly to convolution algebras and abandon points of c-semigroups themselves the future generalizations in the spirit of non-commutative geometry [6] are possible.

We give several examples of c-semigroup now.

**Example 3.4 (Principal)** The main source of c-semigroups is the following construction. Let we have a set  $S$  with an additional structure  $\mathcal{A}$ . Then the set of all mappings from  $S$  into  $S$ , which preserve  $\mathcal{A}$ , forms a right c-semigroup.

**Example 3.5** If one considers in the previous Example only mappings  $S$  onto  $S$  then they give us a definition of groups. Moreover a large set of c-semigroups can be constructed from groups directly. Namely, let  $C$  be a subset of a group  $G$  such that  $(C \cdot C) \subset C$ . Then  $C$  with the multiplication induced from  $G$  will be a c-semigroup. In such a way we obtain the c-semigroup of positive real (natural, rational) numbers with the usual addition. Entire (positive entire) numbers with multiplication also form a c-semigroup. If  $C$  contains the identity  $e$  of  $G$  then  $e$  is a source of  $C$ . Particularly, entire group  $G$  is also a c-semigroup. We should note, that even for groups some of our technique will be new (for example *tokens*), at least up to the author knowledge.

The last example demonstrates that we have many c-semigroups with (left, right) invariant measures such that Fubini's theorem holds. This follows from corresponding constructions for groups.

We sign out the most important for the present paper example from the described family and his alternative realization.

**Proposition 3.6** *The algebra of formal power series in a variable  $t$  is topologically isomorphic to the convolution algebra  $\mathcal{C}(\mathbb{N}_+)$  of continuous functions with point-wise convergence over the c-semigroup of non-negative integers  $\mathbb{N}_+$  with the discrete topology. The subalgebra of polynomials in one variable  $t$  is isomorphic to convolution algebra  $\mathcal{C}_0(\mathbb{N}_+)$ , which is dense in  $\mathcal{C}(\mathbb{N}_+)$ , of compactly supported functions on  $\mathbb{N}_+$ .*

The proof is hardly needed. Note that “puzzling questions” about convergence of formal power series and continuity of functionals on them correspond to the natural topology on  $\mathcal{C}(\mathbb{N}_+)$ .

Less obvious family of examples can be constructed from a structure quite remote (at least at the first glance) from groups.

**Example 3.7** Let  $P$  be a *poset* (i.e., *partially ordered set*) and let  $C$  denote the subset of Cartesian square  $P \times P$ , such that  $(a, b) \in C$  iff  $a \leq b$ ,  $a, b \in P$ . We can define a multiplication on  $C$  by the formula:

$$(a, b)(c, d) = \begin{cases} \text{undefined}, & b \neq c; \\ (a, d), & b = c. \end{cases} \quad (3.6)$$

One can see that  $C$  is a c-set. If  $P$  is locally finite, i.e., for any  $a \leq b$ ,  $a, b \in P$  the number of  $z$  between  $a$  and  $b$  ( $a \leq z \leq b$ ) is finite, then we can define a measure  $d(a, b) = 1$  on  $C$  for any  $(a, b) \in C$ . With such a measure (3.4) defines the correct convolution on  $C$ :

$$h(a, b) = \int_C f(c, d) g([(c, d)^{-1}(a, b)]) d(c, d) = \sum_{a \leq z \leq b} f(a, z) g(z, b). \quad (3.7)$$

The constructed algebra is the fundamental *incidence algebra* in combinatorics [7].

Our present technique can be successfully applied only to c-semigroups, not to c-sets. Thus it is an important observation that *reduced incidence algebra* construction [7, § 4] contracts c-sets to c-semigroups in many important cases (however not in general). Particularly *all combinatorial* reduced incidence algebras listed in Examples 4.5–4.9 of [7] are convolution algebras over c-semigroups. However the incidence coefficients should be better understood.

**Example 3.8** Any *groupoid* (see [22] and references herein) is a c-set.

After given examples, it is reasonable to expect an applicability of c-semigroups to combinatorics. We describe applications in the next Section explicitly.

Now we point out some basic properties of c-semigroups, trying to be as near as possible to their prototype—groups. All results will be states for left c-semigroups. Under agreement (3.5) we have

**Lemma 3.9 (Shift invariance of integrals)** *For any function  $f(c)$  we have*

$$\int_C f(c) dc = \int_C f([a^{-1}b]) db, \quad \forall a \in C. \quad (3.8)$$

Proof follows from the observation that the integrals in both sides of (3.8) is taken over the whole  $C$  and measure  $db$  is invariant. One should be warned that

$$\int_C f(c) dc \neq \int_C f(ab) db$$



for  $c$ -semigroups (unlike for groups) in general as well as Lemma 3.9 is false for  $c$ -sets. Another very useful property (which  $c$ -semigroups possess from the groups) is a connection between linear functional and shift invariant operators. Again unlike in the group case we should make the right choice of shifts.

**Definition 3.10** We define *left shift operator*  $\lambda_a$  on the space of function  $\mathcal{L}_1(C)$  by the formula  $[\lambda_a f](b) = f(ab)$ . A linear operator  $L$  is (left) *shift invariant* if  $S\lambda_a = \lambda_a S$  for all  $a \in C$ . The *augmentation*  $\epsilon$  associated to a right source  $e$  is the linear functional defined by  $\epsilon(f) = f(e)$ .

**Proposition 3.11** *Let  $C$  will be a  $c$ -semigroup with a right source  $e$ . The following three spaces are in one-to-one correspondence:*

1. *The space  $\mathcal{BL}$  of liner functionals on the space of functions  $\mathcal{L}(C)$ ;*
2. *The space  $\mathcal{BS}$  of shift invariant operators;*
3. *The algebra of convolutions  $\mathfrak{C}$  with functions from  $\mathcal{L}^*(C)$ .*

The correspondences are given by the formulas ( $l \in \mathcal{L}^*(C)$ ,  $S \in \mathcal{S}$ ,  $k \in \mathfrak{C}$ ,  $f \in \mathcal{L}(C)$ )

$$\begin{aligned} l &\rightarrow S, \text{ where } [Sf](a) = \langle l, \lambda_a f \rangle \\ S &\rightarrow l, \text{ where } lf = \epsilon(Sf) \\ k &\rightarrow l, \text{ where } lf = \int_C k(c)f(c) dc \\ k &\rightarrow S, \text{ where } Sf(a) = \int_C k(c)f(ac) dc \end{aligned}$$

PROOF. Let we have a shift invariant operator  $S$ . Clearly that its composition  $lf = \epsilon Sf = [Sf](e)$  with the augmentation is a linear functional. By the definition of distributions [9, § III.4] such a linear functional can be represented as the integral  $lk = \int_C k(c)f(c) dc$  with a (probably generalized) function  $k(c)$  from the dual space. For any function  $f$  we have  $f(a) = \lambda_{ae^{-1}} f(e)$  thus by the shift-invariance of  $S$

$$Sf(a) = [\lambda_{ae^{-1}} Sf](e) = [S\lambda_{ae^{-1}} f](e)$$

or by the definition of  $l$  we find  $Sf(a) = \langle l, \lambda_{ae^{-1}} f \rangle$ . The integral form of the last identity is  $Sk(a) = \int_C k(c)f(ac) dc$ , which coincides with (3.1).  $\square$

An essential rôle in all three approaches (operator, Hopf algebras, semantic) to the umbral calculus is played by linear functional (and thus associated linear shift invariant operators). Connections between linear functionals and shifts can be greatly simplified if we are able to express shifts via some “linear coefficients” of functions. The known tool to do that is the Fourier transform. An alternative tool for c-semigroups is presented in next section.

### 3.2 Tokens: Definition and Examples

We formalize identity (2.1)

$$p_n(x + y) = \sum_{k=0}^n p_k(x) p_{n-k}(y).$$

in the following notion, which will be useful in our consideration and seems to be of an interest for a representation theory. The main usage of this notion is  $t$ -transform defined in the next subsection.

**Definition 3.12** Let  $C_1$  and  $C_2$  be two c-semigroups. We will say that a function  $t(c_1, c_2)$  on  $C_1 \times C_2$  is a *token* from  $C_1$  to  $C_2$  if for any  $c'_1 \in C_1$  and any  $c_2, c'_2 \in C_2$  we have

$$\int_{C_1} t(c_1, c_2) t([c_1^{-1} c'_1], c'_2) dc_1 = t(c'_1, c_2 c'_2). \quad (3.9)$$

**REMARK 3.13** We derive this definition “experimentally” from (2.1) (see also Examples bellow). However one can discover it “theoretically” as well. We already know from Proposition 3.11 that a shift invariant operator  $S$  can be represented as the combination of shifts and a linear functional:  $[Sf](a) = \langle l, \lambda_a f \rangle$ . This suggests to represent a function  $f(b)$  as a linear combinations

$$f(b) = \int e(b, c) \hat{f}(c) dc \quad (3.10)$$

of elementary components  $e(b, c)$ , which behave simply under shifts. Here “simply” means *do not destroy linear combination* (3.10).

A way to achieve this is provided by the Fourier transform. Here  $e(b, c) = e^{ibc}$  and the action of a shift  $\lambda_a$  reduces to multiplication:

$$\lambda_a e(b, c) = e(b + a, c) = e^{iac} e(b, c). \quad (3.11)$$

This probably is the simplest solution but not the only possible one. More general transformation of this sort is given by the abstract wavelets (or coherent states):

$$\lambda_a e(b, c) = e(b + a, c) = \int k(a, c, c_1) e(b, c_1) dc_1, \quad (3.12)$$

where we can particularly select  $k(a, c, c_1) = e^{iac}\delta(c - c_1)$  to get the rule of exponents (3.11) and  $k(a, c, c_1) = e(c_1^{-1}c, a)$  to get the token property (3.9). Then shifts act on the functions defined via (3.10) as follows

$$\begin{aligned} \lambda_a f(b) &= \lambda_a \int e(b, c) \hat{f}(c) dc \\ &= \int \int k(a, c, c_1) e(b, c_1) dc_1 \hat{f}(c) dc \\ &= \int e(b, c_1) \left( \int k(a, c, c_1) \hat{f}(c) dc \right) dc_1. \end{aligned}$$

Thus it again looks like (3.10) where the action of a shift reduced to an integral operator on the symbol  $\hat{f}(c)$ . In the case of token shifts affect in the way (3.9), which stand between the extreme simplicity of (3.11) and the almost inaccessible generality of (3.12). This analogy with the Fourier transform will be employed in the next Subsection for the definition of  $t$ -transform.

**REMARK 3.14** Relationships between harmonic analysis (group characters) and token-like structures (polynomial of binomial type) was already pointed in [5]. Note that tokens are a complementary (in some sense) tool for the exponent  $e^{ibc}$ . While both are particularly useful in an investigation of shift invariant operators they work in different ways. The Fourier transform maps shift invariant operators to operators of multiplication.  $t$ -Transform defined in the next subsection by means of tokens maps shift invariant operators on one  $c$ -semigroup to shift invariant operators on another  $c$ -semigroup.

We present examples of tokens within classical objects.

**Example 3.15** Let  $C_1$  be the  $c$ -semigroup of non-negative natural numbers  $\mathbb{N}_+$  and  $C_2$  be the  $c$ -semigroup (a group in fact) of real numbers both equipped by addition. Let  $t(n, x) = p_n(x)$ , where  $p_n(x)$  is a polynomial sequence of binomial type. Then the characteristic property of such a sequence (2.1) is equal to the definition of token (3.9) read from the right to left.

**Example 3.16** We exchange the  $C_1$  and  $C_2$  from the previous Example, they are real numbers and non-negative natural numbers respectively. Let us take a shift invariant operator  $L$ . Any its power  $L^n$  is again a shift invariant operator represented by a convolution with a function  $l(n, x)$  with respect to variable  $x$ . The identity  $L^{n+k} = L^n L^k$  can be expressed in the term of the correspondent kernels of convolutions:

$$\int_{\mathbb{R}} l(n, x) l(k, y - x) dx = l(n + k, y). \quad (3.13)$$

First two examples with their explicit duality generate a hope that our development can be of some interest for combinatorics. But before working out this direction we would like to present some examples of tokens outside combinatorics.

**Example 3.17** Let  $C_1$  be  $\mathbb{R}^n$  and  $C_2$  is  $\mathbb{R}^n \times \mathbb{R}_+$ —the “upper half space” in  $\mathbb{R}^{n+1}$ . For the space of harmonic function in  $C_2 = \mathbb{R}^n \times \mathbb{R}_+$  there is an integral representation over the boundary  $C_1 = \mathbb{R}^n$ :

$$f(v, t) = \int_{C_1} P(u; v, t) f(u) du, \quad u \in C_1, (v, t) \in C_2, v \in \mathbb{R}^n, t \in \mathbb{R}_+.$$

Here  $P(u, v)$  is the celebrated Poisson kernel

$$P(u; v, t) = \frac{2}{|S_n|} \frac{t}{(|u - v|^2 + t^2)^{(n+1)/2}}$$

with the property usually referred as a *semigroup property* [2, Chap. 3, Prob. 1]

$$P(u; v + v', t + t') = \int_{C_1} P(u'; v', t') P(u - u'; v, t) du'.$$

We meet the token in analysis.

**Example 3.18** We preserve the meaning of  $C_1$  and  $C_2$  from the previous Example and define the Weierstrass (or Gauss-Weierstrass) kernel by the formula:

$$W(z; w, \tau) = \frac{1}{(\sqrt{2\pi\tau})^n} e^{-\frac{|z-w|^2}{2\tau}} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{\tau}{2}|u|^2} e^{-(u, z-w)} du,$$

where  $z \in C_1$ ,  $(w, \tau) \in C_2$ . Function  $W(z; w, \tau)$  is the fundamental solution to the heat equation [8, § 2.3]. We again have [2, Chap. 3, Prob. 1]

$$W(z; w + w', \tau + \tau') = \int_{C_1} W(z'; w', \tau') W(z - z'; w, \tau) dz.$$

Thus we again meet a token.

Two last Examples open a huge list of integral kernels [2] which are tokens of analysis. We will see the rôle of tokens for analytic function theory and reproducing kernels later.

We finish the subsection by the following elementary property of token.

**Lemma 3.19** *Let  $c$ -semigroup  $C_2$  has a left source  $e_2$ . Then  $k(c_1) = t(c_1, e_2)$  has the reproducing property*

$$t(c'_1, c_2) = \int_{C_1} k(c_1) t([c_1^{-1}c'_1], [e_2^{-1}c_2]) dc_1 \quad \forall c'_1 \in C_1, c_2 \in C_2.$$

Particularly if  $e_2 \in C_2$  is a left unit then:

$$t(c'_1, c_2) = \int_{C_1} k(c_1) t([c_1^{-1}c'_1], c_2) dc_1 \quad \forall c'_1 \in C_1, c_2 \in C_2.$$

PROOF. We apply the characteristic property of a token (3.9) for particularly selected elements in  $C$ :

$$\begin{aligned} t(c'_1, c_2) &= t(c'_1, e_2[e_2^{-1}c_2]) \\ &= \int_{C_1} t(c_1, e_2) t([c_1^{-1}c'_1], [e_2^{-1}c_2]) dc_1 \\ &= \int_{C_1} k(c_1) t([c_1^{-1}c'_1], [e_2^{-1}c_2]) dc_1. \end{aligned}$$

□

### 3.3 t-Transform

We would like now to introduce a transformation associated with tokens (t-transform). t-Transform is similar to the Fourier and wavelet transforms. The main difference—we do not insist that our transformation maps convolutions to multiplications. We will be much more modest—our transformation maps convolutions to other convolutions, eventually more simple or appropriate.

Having a function  $\hat{f}(c_1)$  on  $C_1$  we can consider it as coefficients (or a *symbol*) and define the function  $f(c_2)$  on  $C_2$  by means of the token  $t(c_1, c_2)$ :

$$f(c_2) = \int_{C_1} \hat{f}(c_1) t(c_1, c_2) dc_1. \quad (3.14)$$

We will say that  $f(c_2) \in \mathcal{A}(C_2)$  if in (3.14)  $\hat{f}(c_1) \in \mathcal{C}_0(C_1)$  the space of continuous functions with a compact support. So we can consider the token

$t(c_1, c_2)$  as a kernel of integral transform  $\mathcal{C}_0(C_1) \rightarrow \mathcal{A}(C_2)$ . The use of the notation reserved for the Fourier transform ( $\hat{\phantom{x}}$ —“hat”) is justified (at least partially) by Remark 3.13. We would like to mention the following

**Corollary 3.20** *The reproducing property of  $k(c_1)$  from Lemma 3.19 can be extended by linearity to  $\mathcal{A}(C_1)$ :*

$$f(c'_1) = \int_{C_1} k(c_1) f([c_1^{-1}c'_1]) dc_1.$$

**Example 3.21** We continuing with a polynomial sequence of binomial type  $p_k(x)$ , which was presented as a token  $p(k, x)$  in Example 3.15. Having few numbers  $\hat{a}_1, \hat{a}_2, \dots$ , which are interpreted as a function  $\hat{a}(k)$  in  $\mathcal{C}_0(\mathbb{N}_+)$ , we can construct a polynomial

$$a(x) = \int_{\mathbb{N}_+} \hat{a}(k) p(k, x) dk = \sum_k \hat{a}_k p_k(x),$$

that is a function on  $\mathbb{R}$ .

**Example 3.22** Let  $\delta^{(k)}(x)$  be a token from  $\mathbb{R}$  to  $\mathbb{N}_+$  (Example 3.16). Then for a smooth at point 0 function  $f(x)$  the integral transformation

$$f^{(k)}(0) = \int_{\mathbb{R}} f(x) \delta^{(k)}(x) dx$$

produces a function on  $\mathbb{N}_+$ .

**Example 3.23** Let a real valued function  $f(u)$  is defined on  $C_1 = \mathbb{R}^n$ . Then we can define the function  $Pf(v, t)$  on  $C_2 = \mathbb{R}^n \times \mathbb{R}_+$  by means of the Poisson integral:

$$\begin{aligned} Pf(v, t) &= \int_{C_1} P(u; v, t) f(u) du \\ &= \frac{2}{|S_n|} \int_{\mathbb{R}^n} \frac{t}{(|u - v|^2 + t^2)^{(n+1)/2}} f(u) du. \end{aligned}$$

The image is a harmonic function solving the Dirichlet problem [20, Appendix, § 2], [8, § 2.2.4].

**Example 3.24** Similarly we define the Weierstrass transform:

$$\begin{aligned} Wf(w, \tau) &= \int_{C_1} W(z; w, \tau) f(z) dz \\ &= \frac{1}{(\sqrt{2\pi\tau})^n} \int_{\mathbb{R}^n} e^{-\frac{|z-w|^2}{2\tau}} f(z) dz \end{aligned}$$

The image is a function satisfying the heat equation.

Let  $K$  be a convolution on  $C_2$  with a kernel  $k(c_2)$ . Then it acts on function  $f(c_2)$  defined by (3.14) as follows

$$\begin{aligned} [Kf](c_2) &= \int_{C_2} k(c'_2) f(c_2 c'_2) dc'_2 \\ &= \int_{C_2} k(c'_2) \int_{C_1} \hat{f}(c'_1) t(c'_1, c_2 c'_2) dc'_1 dc'_2 \\ &= \int_{C_2} k(c'_2) \int_{C_1} \hat{f}(c'_1) \int_{C_1} t(c_1, c_2) t([c_1^{-1} c'_1], c'_2) dc_1 dc'_1 dc'_2 \quad (3.15) \end{aligned}$$

$$\begin{aligned} &= \int_{C_1} \int_{C_1} \int_{C_2} k(c'_2) t([c_1^{-1} c'_1], c'_2) dc'_2 \hat{f}(c'_1) dc'_1 t(c_1, c_2) dc_1 \\ &= \int_{C_1} \int_{C_1} \int_{C_2} k(c'_2) t(c'_1, c'_2) dc'_2 \hat{f}(c_1 c'_1) dc'_1 t(c_1, c_2) dc_1 \quad (3.16) \end{aligned}$$

$$= \int_{C_1} \left( \int_{C_1} [\mathcal{T}k](c'_1) \hat{f}(c_1 c'_1) dc'_1 \right) t(c_1, c_2) dc_1 \quad (3.17)$$

where

$$[\mathcal{T}k](c'_1) = \int_{C_2} k(c'_2) t(c'_1, c'_2) dc'_2. \quad (3.18)$$

We use in (3.15) the characteristic property of the tokens (3.9) and the change of variables  $c'_1 = [c_1^{-1} c'_1]$  in (3.16). Note that big braces in (3.17) contain a convolution on  $C_1$  with the kernel (3.18). So transformation (3.14) maps convolutions on  $C_2$  to the convolutions on  $C_1$ . The mapping deserves a special name.

**Definition 3.25**  $t$ -Transform is a linear integral transformation  $\mathfrak{C}(C_2) \rightarrow \mathfrak{C}(C_1)$  defined on kernels by the formula:

$$[\mathcal{T}k](c_1) = \int_{C_2} t(c_1, c_2) k(c_2) dc_2. \quad (3.19)$$

According to Lemma 3.9 we have

$$[\mathcal{T}k](c_1) = \int_{C_2} t(c_1, [c_2^{-1} c'_2]) k([c_2^{-1} c'_2]) dc'_2, \quad \forall c_2 \in C_2. \quad (3.20)$$

The  $t$ -transform is more than just a linear map:

**Theorem 3.26**  *$t$ -transform is an algebra homomorphism of the convolution algebra  $\mathfrak{C}(C_2)$  to the convolution algebra  $\mathfrak{C}(C_1)$ , namely*

$$[\mathcal{T}(k_1 * k_2)] = [\mathcal{T}k_1] * [\mathcal{T}k_2], \quad (3.21)$$

where  $*$  in the left-hand side denotes the composition of convolutions on  $C_2$  and in the right-hand side—on  $C_1$ .

PROOF. We silently assume that conditions of Fubini's theorem fulfill. It turns to be that Fubini's theorem will be our main tool, which is used in almost all proofs. Thereafter we assume that it fulfills through entire paper. We have:

$$\begin{aligned} [\mathcal{T}(k_2 * k_1)](c'_1) &= \int_{C_2} t(c'_1, c'_2) \int_{C_2} k_2(c_2) k_1([c_2^{-1}c'_2]) dc_2 dc'_2 \\ &= \int_{C_2} \int_{C_2} \int_{C_1} t(c_1, c_2) t([c_1^{-1}c'_1], [c_2^{-1}c'_2]) dc_1 \\ &\quad \times k_2(c_2) k_1([c_2^{-1}c'_2]) dc_2 dc'_2 \\ &= \int_{C_1} \int_{C_2} t(c_1, c_2) k_2(c_2) dc_2 \\ &\quad \times \int_{C_2} t([c_1^{-1}c'_1], [c_2^{-1}c'_2]) k_1([c_2^{-1}c'_2]) dc'_2 dc_1 \\ &= \int_{C_1} [\mathcal{T}k_2](c_1) [\mathcal{T}k_1]([c_1^{-1}c'_1]) dc_1. \end{aligned} \quad (3.22)$$

$$(3.23)$$

Of course, we use in transformation (3.22) the characteristic property of the tokens (3.9) and conclusion (3.23) is based on (3.20).  $\square$

Knowing the connection between shift invariant operators and convolutions (Proposition 3.11) we derive

**Corollary 3.27**  *$t$ -Transform maps left shift invariant operators on  $C_2$  to left shift invariant operators on  $C_1$ .*

## 4 Shift Invariant Operators

In this Section we present some results in terms of  $c$ -semigroups, tokens and  $t$ -transform and illustrate by examples from combinatorics and analysis.



## 4.1 Delta Families and Basic Distributions

This Subsection follows the paper [17] with appropriate modifications.

**Definition 4.1** Let  $C_1$  and  $C_2$  be two c-semigroups; let  $e_1$  be a right source of  $C_1$ . We will say that linear shift invariant operators  $Q(c_1)$ ,  $c_1 \in C_1$  on  $\mathcal{A}(C_2)$  over c-semigroup  $C_2$  form a *delta family* if

1. for a token  $t(c_1, c_2)$

$$Q(c_1) t(c_1, c_2) = t(e_1, c_2) \quad \text{for all } c_2 \in C_2.$$

2. the semigroup property  $Q(c_1) Q(c'_1) = Q(c_1 c'_1)$  holds.

**Example 4.2** Let  $C_1$  and  $C_2$  be as in Examples 3.15, 3.21, and 5.3. Let also  $t(n, x) = \frac{x^n}{n!}$ . Then the family of positive integer powers of the derivative  $Q(n) = D^n$  forms a delta. Other deltas for the given  $C_1$  and  $C_2$  are listed in [12, § 3]. They are positive integer powers of difference, backward difference, central difference, Laguerre, and Abel operators.

**Example 4.3** For the Poisson kernel from Example 3.17 we can introduce delta family as Euclidean shift operators:

$$Q(x) f(v, t) = f(v + x, t), \quad x \in \mathbb{R}^n.$$

**Proposition 4.4** Let  $q(c_1, c_2)$  be the kernel corresponding to a delta family  $Q(c_1)$  as a convolution over  $C_2$ . Then  $q(c_1, c_2)$  is a token from  $C_2$  to  $C_1$ . Namely

$$q(c_1 c'_1, c'_2) = \int_{C_2} q(c_1, c_2) q(c'_1, [c_2^{-1} c'_2]) dc_2.$$

PROOF. This is a simple restatement in terms of kernels of the semigroup property  $Q(c_1) Q(c'_1) = Q(c_1 c'_1)$ .  $\square$

**Definition 4.5** The function  $t(c_1, c_2)$  is called the *basic distribution* for a delta family  $Q(c_1)$  if:

1.  $t(c_1, e_2) = k(c_1)$  has the reproducing property from Lemma 3.19 over  $C_1$  (particularly if  $k(c_1) = \delta(c_1)$ );
2.  $Q(c_1) t(c'_1, c_2) = t([c_1^{-1} c'_1], c_2)$ .

**Example 4.6** For the classic umbral calculus this conditions turn to be  $p_0(x) = 1$ ;  $p_n(0) = 0$ ,  $n > 0$ ;  $Qp_n(x) = p_{n-1}(x)$ .

**Example 4.7** For the Poisson integral the reproducing property can be obtained if we consider a generalized Hardy space  $\mathcal{H}_2(\mathbb{R}_+^{n+1})$  of  $\mathcal{L}_2$ -integrable functions on  $\mathbb{R}^n$ , which are limit value of harmonic functions in the upper half space  $\mathbb{R}^n \times \mathbb{R}_+$ . Then kernel  $k(u - u')$  is defined as a distribution given by the limit of integral on  $\mathcal{H}_2(\mathbb{R}^n)$ :

$$\begin{aligned} f(u) &= \int_{\mathbb{R}^n} k(u - u') f(u') du' \\ &= \frac{2}{|S_n|} \lim_{(v,t) \rightarrow (0,0)} \int_{\mathbb{R}^n} \frac{t}{(|(u - u') - v|^2 + t^2)^{(n+1)/2}} f(u') du'. \end{aligned}$$

This follows from the boundary property of the Poisson integral [20, Appendix, § 2], [8, § 2.2.4].

**Theorem 4.8** Let  $C_1$  and  $C_2$  be  $c$ -semigroups.

1. If  $t(c_1, c_2)$  is a basic distribution for some delta family  $Q(c_1)$ ,  $c_1 \in C_1$ , then  $t(c_1, c_2)$  is a token from  $C_1$  to  $C_2$ .
2. If  $t(c_1, c_2)$  is a token from  $C_1$  to  $C_2$ , then it is a basic distribution for some delta family  $Q(c_1)$ ,  $c_1 \in C_1$ .

PROOF. Let  $t(c_1, c_2)$  be a basic distribution for  $Q(c_1)$  and let  $q(c_2, c_1)$  be a kernel of operator  $Q(c_1)$  as a convolution on  $C_2$ . This means that

$$\langle q(c_2, c_1), t(c'_1, c_2) \rangle_{c_2} = k([c_1^{-1}c'_1])$$

has a reproducing property due to Lemma 3.19 and Corollary 3.20. Thus we can trivially express  $t(c'_1, c'_2)$  as the integral

$$t(c'_1, c'_2) = \int_{C_1} t(c_1, c'_2) \langle q(c_2, c_1), t(c'_1, c_2) \rangle_{c_2} dc_1.$$

By linearity we have a similar expression for any function in  $\mathcal{A}(C_2)$ :

$$f(c'_2) = \int_{C_1} t(c_1, c'_2) \langle q(c_2, c_1), f(c_2) \rangle_{c_2} dc_1.$$

In particular

$$t(c'_1, c''_2 c'_2) = \int_{C_1} t(c_1, c'_2) \langle q(c_2, c_1), t(c'_1, c''_2 c'_2) \rangle_{c_2} dc_1.$$

But

$$\begin{aligned}
 \langle q(c_2, c_1), t(c'_1, c''_2 c_2) \rangle_{c_2} &= Q(c_1) \lambda_{c''_2} t(c'_1, c_2) \mid_{c_2=e_2} \\
 &= \lambda_{c''_2} Q(c_1) t(c'_1, c_2) \mid_{c_2=e_2} \\
 &= \lambda_{c''_2} t([c_1^{-1} c'_1], c_2) \mid_{c_2=e_2} \\
 &= t([c_1^{-1} c'_1], c''_2),
 \end{aligned}$$

and therefore

$$t(c'_1, c''_2 c'_2) = \int_{C_1} t(c_1, c'_2) t([c_1^{-1} c'_1], c''_2) dc_1;$$

that is, the distribution  $t(c_1, c_2)$  is a token.

Suppose now that  $t(c_1, c_2)$  is a token. We define a family of operators  $Q(c_1)$  on  $\mathcal{A}(C_2)$  by identities

$$Q(c_1) t(c'_1, c_2) = t([c_1^{-1} c'_1], c_2)$$

and extending to whole  $\mathcal{A}(C_2)$  by the linearity. The semigroup property  $Q(c_1)Q(c_2) = Q(c_1 c_2)$  follows automatically. The main point is to show that  $Q(c_1)$  are shift invariant. We may trivially rewrite the characteristic property of token as

$$t(c'_1, c'_2 c_2) = \int_{C_1} t(c_1, c'_2) Q(c_1) t(c'_1, c_2) dc_1,$$

which can be extended by linearity to any function in  $\mathcal{A}(C_2)$ :

$$f(c'_2 c_2) = \int_{C_1} t(c_1, c'_2) Q(c_1) f(c_2) dc_1,$$

Now replace  $f$  by  $Q(c'_1)f$

$$Q(c'_1)f(c'_2 c_2) = \int_{C_1} t(c_1, c'_2) Q(c'_1 c_1) f(c_2) dc_1,$$

But the left-hand side of the previous identity is nothing else as  $[\lambda_{c'_2} Q(c'_1)f](c_2)$  and the right-hand side is

$$\begin{aligned}
 \int_{C_1} t(c_1, c'_2) Q(c'_1 c_1) f(c_2) dc_1 &= Q(c'_1) \left( \int_{C_1} t(c_1, c'_2) Q(c_1) f(c_2) dc_1 \right) \\
 &= Q(c'_1) (f(c'_2 c_2)) = [Q(c'_1) \lambda_{c'_2} f](c_2),
 \end{aligned}$$

i.e.,  $Q$  is shift invariant.  $\square$

The following Lemma is obvious.

**Lemma 4.9** *Let  $q(c_2, c_1)$  be the kernel of  $Q(c_1)$  as a convolution on  $C_2$ , then the  $t$ -transform for the token  $q(c_2, c_1)$  is a right inverse operator for  $t$ -transform (3.19) with respect to  $t(c_1, c_2)$ .*

The above Lemmas justify the following Definition.

**Definition 4.10** Let  $t(c_1, c_2)$  be the basic distribution for a delta family  $Q(c_1)$ . Then we call  $t(c_1, c_2)$ , which is a token by Theorem 4.8, and the kernel  $q(c_2, c_1)$  of  $Q(c_1)$ , which is a token by Proposition 4.4, *dual tokens*.

**Example 4.11** The polynomials  $t(n, x) = x^n/n!$  and distributions  $q(n, x) = \delta^{(n)}(x)$  (which are kernels of derivatives operators  $D^{(k)}$ ) are canonical dual tokens.

As the reader may see in Theorem 4.8 we almost do not change proofs of [17, Theorem 1, § 2]. So we give our version of next two results without proofs.

**Theorem 4.12 (First Expansion Theorem)** *Let  $S$  be a shift invariant operator on  $\mathcal{A}(C_2)$  and let  $Q(c_1)$ ,  $c_1 \in C_1$  be a delta family with basic distribution  $t(c_1, c_2)$ . Then*

$$S = \int_{C_1} a(c_1) Q(c_1) dc_1$$

with

$$a(c_1) = \langle s(c_2), t(c_1, c_2) \rangle_{c_2} = St(c_1, c_2) |_{c_2=e_2} .$$

**Theorem 4.13 (Isomorphism Theorem)** *Let  $Q(c_1)$ ,  $c_1 \in C_1$  be a delta family with basic distribution  $t(c_1, c_2)$ . Then mapping which carries a shift invariant operator  $S$  to a function*

$$s(c_2) = \int_{C_1} a(c_1) q(c_2, c_1) dc_1$$

with

$$a(c_1) = \langle s(c_2), t(c_1, c_2) \rangle_{c_2} = St(c_1, c_2) |_{c_2=e_2} .$$

*is an isomorphism of operator algebra to a convolution algebra on  $C_2$ .*

**Example 4.14** For a sequence of binomial type  $p_n(x)$  and the associated delta operator  $Q$  we can decompose a shift invariant operator  $S$  on the space of polynomials as

$$S = \sum_{k=0}^{\infty} a_k Q^k, \quad \text{where} \quad a_k = [Sp_k(x)]_{x=0}.$$

**Example 4.15** Let a shift invariant operator  $S$  on  $\mathcal{H}_2(\mathbb{R}_+^{n+1})$  is given by a kernel  $s(v, t)$ :

$$[Sf](v, t) = \int_{\mathbb{R}_+^{n+1}} s(v', t') f(v + v', t + t') dv' dt'.$$

Then we can represent  $S$  by means of the delta family  $Q(u)$ ,  $u \in \mathbb{R}^n$  from Example 4.3 as follows:

$$S = \int_{\mathbb{R}^n} a(u) Q(u) du$$

where

$$a(u) = \int_{\mathbb{R}_+^{n+1}} s(v, t) P(u; v, t) dv dt.$$

And a similar representation is true for the Weierstrass kernel.

The following result has a somewhat cumbersome formulation and a completely evident proof. In the classic umbral calculus case the formulation turns to be very natural, but proof more hidden (see next example).

**Theorem 4.16** Let  $t_1(c_1, c_2)$  and  $t_2(c_1, c_2)$  be two tokens from  $C_1$  to  $C_2$ . Let  $c$ -semigroup  $C_1$  be equipped with an partial order relation  $>$ . Moreover let

$$t_2(c_1, c_2) = \int_{C_1} a(c_1, c'_1) t_1(c'_1, c_2) dc'_1 \quad (4.1)$$

where  $a(c_1, c_1) = 0$  whenever  $c_1 > c'_1$ . Let  $Q_2(c_1)$  be a delta family associated to  $t_2(c_1, c_2)$ . Let  $\sup_{t_1}(f) \in C_1$  denotes the support of  $t$ -transform of a function  $f(c_2)$  with respect to  $t_1(c_1, c_2)$ . Then for any function  $f(c_2)$  and  $c_1 \in C_1$ :

$$\lambda_{c_1}(\sup_{t_1}(Q(c_1)f)) \subset \sup_{t_1}(f).$$

PROOF. This is a direct consequence of shift invariance of a delta family  $Q(c_1)$  as operators on  $C_2$  and Corollary 3.27.  $\square$

**Example 4.17** C-semigroup  $\mathbb{N}_+$  of natural numbers is naturally ordered. By the very definition [17, § 3] for any polynomial sequence of binomial type  $p(k, x)$  is exactly of degree  $k$  for all  $k$ . Thus for any two such sequences  $p_1(k, x)$  and  $p_2(k', x)$  we have

$$p_1(k, x) = \sum_{k'=1}^k a(k, k') p_2(k', x),$$

i.e., (4.1) is satisfied. Let now  $Q(k) = Q^k$  be the delta family associated to  $p_1(k, x)$ , then by Theorem 4.16 polynomial  $Q(k)p_2(k', x)$  is of degree  $k' - k$ . This gives [17, Propositions 1 and 2, § 2].

Generally speaking an order on a set of combinatorial number allows to consider some recurrence relations, which express a combinatorial number via smaller numbers of the same kind. We will return to this subject in Subsection 4.3 in a connection with generating functions.

## 4.2 Generating Functions and Umbral Functionals

It is known that generating functions is a powerful tool in combinatorics. We try to interpret the notion of generating functions in our terms. Particularly we will treat Examples 4.5–4.9 from [7] accordingly.

**Definition 4.18** Let function  $f(c_1)$  represents a set of combinatorial quantities indexed by points of c-semigroup  $C_1$  (a *combinatorial function* for short). Let  $C_2$  be another c-semigroup and  $t(c_1, c_2)$  be a token between them. The t-transform

$$\hat{f}(c_2) = \int_{C_1} f(c_1) t(c_1, c_2) dc_1$$

of  $f(c_1)$  is called *generating function* for  $f(c_1)$  (with respect to  $C_2$  and  $t(c_1, c_2)$ )

Following [14] an exceedingly useful method for defining a function  $f(c_1)$  is to apply an umbral linear functional  $l$  to a token  $p(c_1, c_2)$ :

$$f(c_1) = \langle l, p(c_1, c_2) \rangle := \int_{C_2} l(c_2) p(c_1, c_2) dc_2. \quad (4.2)$$

If we now consider the generating function  $\hat{f}(c_2)$  with respect to the token  $q(c_2, c_1)$ , which is dual to  $p(c_1, c_2)$  then we will found that

$$\hat{f}(c_2) = \int_{C_1} f(c_1) q(c_2, c_1) dc_1$$

$$\begin{aligned}
&= \int_{C_1} \int_{C_2} l(c'_2) p(c_1, c'_2) dc'_2 q(c_2, c_1) dc_1 \\
&= \int_{C_2} l(c'_2) \int_{C_1} p(c_1, c'_2) q(c_2, c_1) dc_1 dc'_2 \\
&= \int_{C_2} l(c'_2) \delta(c'_2, c_2) dc'_2 \\
&= l(c_2)
\end{aligned}$$

So we found the following simple connection between umbral functionals and generating functions

**Theorem 4.19** *The generating function  $\hat{f}(c_2)$  for a combinatorial function  $f(c_1)$  with respect to a token  $q(c_2, c_1)$  is the kernel of umbral functional for  $f(c_1)$  with respect to the token  $p(c_1, c_2)$  dual to token  $q(c_2, c_1)$ .*

**Example 4.20** Let  $\{f_n\}_{k=0}^{\infty}$  be the sequence of the Fibonacci numbers. Let  $F(x)$  be the generating function associated to  $\{f_n\}_{k=0}^{\infty}$  by the token  $q(k, x) = \delta^{(k)}(x)$ :

$$\begin{aligned}
F(x) &= \sum_{k=0}^{\infty} f_n \delta^{(k)}(x) \quad \left[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 - (i\xi) - (i\xi)^2} e^{-i\xi x} d\xi \right] \\
&= \frac{1}{\sqrt{5}} e^{\frac{1-\sqrt{5}}{2}x} \left( 1 + (e^{\sqrt{5}x} - 1) \chi(-x) \right)
\end{aligned}$$

where  $\chi(x)$  is the Heaviside function:

$$\chi(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0. \end{cases}$$

Then  $F(x)$  is the kernel of an umbral functional, which generates the sequence  $\{f_n\}_{k=0}^{\infty}$  from polynomial sequence  $t(n, x) = x^n/n!$  which is dual to  $q(k, x) = \delta^{(k)}(x)$  (Example 4.11):

$$f_n = \int_{-\infty}^{\infty} F(x) \frac{x^n}{n!} dx.$$

In this way we obtain a constructive solution for the classic moment problem [1].

### 4.3 Recurrence Operators and Generating Functions

We are looking for reasons why some generating functions are better suited to handle given combinatorial functions than other. The answer is given by their connections with recurrence operators.

We will speak on *combinatorial functions* here that are functions defined on a  $c$ -semigroups. Different numbers (Bell, Fibonacci, Stirling, etc.) known in combinatorics are, of course, combinatorial functions defined on  $c$ -semigroup  $\mathbb{N}^{[+]}$ .

**Definition 4.21** An operator  $R$  on  $C_1$  is said to be a *recurrence operator* with respect to a token  $t(c_1, c_2)$  for a combinatorial function  $f(c_1)$  if

$$[Rf](c_1) = t(c_1, e_2). \quad (4.3)$$

Particularly  $[Rf](c_1)$  has a reproducing property from Lemma 3.19.

If we consider an algebra of convolutions of combinatorial functions then by the general property of  $t$ -transform *it is isomorphic to convolution algebra of their generating function*, which occasionally can be isomorphic to an algebra with multiplication (multiplication of formal power series, exponential power series, Dirichlet series, etc.). This was already observed by other means in [7].

The following Theorem connects generating functions and recurrent operators.

**Theorem 4.22** Let  $C_1$  and  $C_2$  be  $c$ -semigroups with a token  $t(c_1, c_2)$  between them and  $q(c_2, c_1)$  be its dual token. Let  $f(c_1)$  be a combinatorial function with a corresponding recurrence operator  $R$ , which is defined by its kernel  $r(c_1, c'_1)$

$$[Rf](c_1) = \int_{C_1} r(c_1, c'_1) f(c'_1) dc'_1$$

Then the generating function  $\hat{f}(c_1)$  satisfies the equation

$$[\tilde{R}\hat{f}](c_2) = t(e_1, c_2), \quad (4.4)$$

where  $\tilde{R}$  is defined by the kernel

$$\tilde{r}(c_2, c'_2) = \int_{C_1} \int_{C_1} t(c'_1, c_2) r(c'_1, c'_1) q(c'_1, c'_2) dc'_1 dc'_2. \quad (4.5)$$

PROOF. We start from an observation that

$$\int_{C_2} \tilde{r}(c_2, c'_2) t(c_1, c'_2) dc'_2 = \int_{C_1} t(c'_1, c_2) r(c'_1, c_1) dc'_1, \quad (4.6)$$



which follows from application to both sides of (4.5) the integral operator with kernel  $t(c_1, c'_2)$  and the identity

$$\int_{C_2} q(c'_1, c'_2) t(c'_2, c_1) dc'_2 = \delta(c'_1, c_1).$$

Then

$$\begin{aligned} [\tilde{R}\hat{f}](c_2) &= \int_{C_2} \tilde{r}(c_2, c'_2) \hat{f}(c'_2) dc_2 \\ &= \int_{C_2} \hat{r}(c_2, c'_2) \int_{C_1} f(c_1) t(c_1, c'_2) dc_1 dc_2 \\ &= \int_{C_1} \int_{C_2} \hat{r}(c_2, c'_2) t(c_1, c'_2) dc_2 f(c_1) dc_1 \\ &= \int_{C_1} \int_{C_1} t(c''_1, c_2) r(c''_1, c_1) dc''_1 f(c_1) dc_1 \quad (4.7) \\ &= \int_{C_1} t(c''_1, c_2) \int_{C_1} r(c''_1, c_1) f(c_1) dc_1 dc''_1 \\ &= \int_{C_1} t(c''_1, c_2) \delta(c''_1) dc''_1 \\ &= t(e_1, c_2) \end{aligned}$$

where we deduce (4.7) from (4.6).  $\square$

In fact, our calculation just says that  $\tilde{R}g = T\delta$  for  $g = Tf$  and operator  $\tilde{R} = TRT^{-1}$  if  $Rf = \delta$ . A touch of non-triviality appears when we use special properties of functions  $t(c_1, c_2)$  and  $q(c_2, c_1)$  be a pair of dual tokens:

**Corollary 4.23** *If under assumptions of Theorem 4.22 the operator  $R$  is shift invariant on  $C_1$ , i.e. has a kernel of the form  $r(c_1, c'_1) = r(c_1^{-1}c'_1)$ , then the operator  $\tilde{R}$  is also shift invariant on  $C_2$  with a kernel of the form  $\hat{r}(c_2, c'_2) = r(c_2'^{-1}c_2)$ .*

PROOF. This follows from the property of t-transform to map a shift invariant operator on  $C_1$  to a shift invariant operator on  $C_2$ .  $\square$

REMARK 4.24 The above corollary clearly indicates what type of generating functions is reasonable: *a simple generating function can be constructed with respect to a token  $t(c_1, c_2)$  such that the recurrence operator  $R$  is shift-invariant with respect to it.*

As it usually occurs a simple fact may have interesting realizations.

**Example 4.25** Let  $f_n$  be the Fibonacci numbers. Their known recursion  $f_n = f_{n-1} + f_{n-2}$  should be stated in a more accurate way with the help of agreement (3.5) as follows:

$$f_n - f_{n-1} - f_{n-2} = \delta_{n,0}, \quad n \geq 0.$$

Then the recurrence operator is given by  $R = I - S - S^2$ , where  $S$  is a backward shift  $Sf(n) = f(n-1)$  on  $\mathbb{N}_+$ , particularly  $R$  is *shift invariant* with respect to token  $t(x, n) = x^n$ . The corresponding kernel is

$$r(n, i) = \delta_{n,i} - \delta_{n-1,i} - \delta_{n-2,i},$$

which is a function of  $n - i$ . For the pair of dual tokens  $t(x, n) = x^n$  and  $q(i, y) = \delta^{(i)}(y)/i!$  we obtain a transformed kernel:

$$\begin{aligned} \tilde{r}(x, y) &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} t(x, n) r(n, i) q(i, y) \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} x^n (\delta_{n,i} - \delta_{n-1,i} - \delta_{n-2,i}) \frac{\delta^{(i)}(y)}{i!} \\ &= \sum_{n=0}^{\infty} x^n (1 - x - x^2) \frac{\delta^{(n)}(y)}{n!} \\ &= (1 - x - x^2) \sum_{n=0}^{\infty} x^n \frac{\delta^{(n)}(y)}{n!} \end{aligned} \tag{4.8}$$

From the Taylor expansion the sum in (4.8) obviously represents an integral kernel of the identity operator, therefore the transformation  $\tilde{R}$  is the operator of multiplication by  $1 - x - x^2$  and the generating function  $\hat{f}(x)$  should satisfy equation

$$\tilde{R}f(x) = (1 - x - x^2)\hat{f}(x) = x^0 = 1.$$

Thus

$$\hat{f}(x) = \sum_{n=0}^{\infty} f_n x^n = \frac{1}{1 - x - x^2}$$

Note that operator  $\tilde{R}$  of multiplication by  $(1 - x - x^2)$  is “*shift invariant*”—it commutes with the operator of multiplication by  $x$  (both these operators are certain convolutions under Fourier transform).

An attempt to construct a transformation  $\tilde{R}$  with respect to a similar pair of dual tokens  $t_1(x, n) = x^n/n!$  and  $q_1(i, y) = \delta^{(i)}(y)$  will not enjoy the above simplicity.

**Example 4.26** Let  $B_n$  be the Bell numbers [14] or [18, § 3]. The known recursion for them [18, § 3] can again be restated as

$$B_n - \sum_{k=0}^{n-1} \binom{n-1}{k} B_k = \delta_{n,0}, \quad n \geq 0. \quad (4.9)$$

Such a recurrence identity is *shift invariant* with respect to  $W$  defined by the identity  $Wf(n) = f(n-1)/n$  on  $\mathbb{N}_+$ . In other words it is *shift invariant* with respect to token  $t(x, n) = x^n/n!$ . The kernel  $r(n, i)$  corresponding to recurrence operator  $R$  in (4.9) is

$$r(n, i) = \delta_{n,i} - \sum_{k=0}^{n-1} \binom{n}{k} \delta_{k,i}.$$

We apply a transformation with respect to a pair of dual tokens  $t(x, n) = x^n/n!$  and  $q(i, y) = \delta^{(i)}(y)$ :

$$\begin{aligned} \tilde{r}(x, y) &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} t(x, n) r(n, i) q(i, y) \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{x^n}{n!} \left( \delta_{n,i} - \sum_{k=0}^{n-1} \binom{n-1}{k} \delta_{k,i} \right) \delta^{(i)}(y) \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{x^n}{n!} \delta_{n,i} \delta^{(i)}(y) - \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n-1} \binom{n-1}{k} \delta_{k,i} \delta^{(i)}(y) \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \delta^{(n)}(y) - \sum_{n=0}^{\infty} \sum_{i=0}^{n-1} \frac{x^n}{n!} \frac{(n-1)!}{i! (n-1-i)!} \delta^{(i)}(y) \\ &= \sum_{n=0}^{\infty} x^n \frac{\delta^{(n)}(y)}{n!} - \sum_{n=0}^{\infty} \sum_{i=0}^{n-1} \frac{1}{n} x^n \frac{\delta^{(i)}(y)}{i! (n-1-i)!} \\ &= \sum_{n=0}^{\infty} x^n \frac{\delta^{(n)}(y)}{n!} - \sum_{n=0}^{\infty} \frac{x}{n} \sum_{i=0}^{n-1} \frac{x^{n-i-1}}{(n-1-i)!} \frac{x^i \delta^{(i)}(y)}{i!} \end{aligned} \quad (4.10)$$

One verifies that an integration of a function  $f(y)$  with the sum

$$\sum_{i=0}^{n-1} \frac{x^{n-i-1}}{(n-1-i)!} \frac{x^i \delta^{(i)}(y)}{i!}$$

in (4.10) produces the  $(n-1)$ -th term in the Taylor expansion of the product  $e^x f(x)$ . Thus the entire expression in (4.10) is a kernel of the operator:

$$\tilde{R} = I - \int e^x,$$

where  $I$  is the identity operator and  $\int$  is the operator of the anti-derivation fixed by the condition  $[\int g](0) = 0$ , particularly  $\int x^n/n! = x^{n+1}/(n+1)!$ . So generating function  $\hat{b}(x)$  should satisfy the equation  $(I - \int e^x)\hat{b}(x) = x^0 = 1$ . Taking the derivative from both sides we found the differential equation  $(D - e^x)\hat{b}(x) = 0$ . With the obvious initial condition  $\hat{b}(0) = 1$  it determines that  $\hat{b}(x) = \exp(\exp(x) - 1)$ .

One may interpret (4.3) as if  $f(c_1)$  is a fundamental solution to operator  $R$ . This is indeed a right way of thinking if one can found a group action such that  $R$  is a shift invariant. Then the convolution  $h(c_1) = [g * f](c_1)$  will give a solution to the equation  $[Rh](c_1) = g(c_1)$ .

## 5 Models for the Umbral Calculus

We present realizations (models) for different descriptions of the umbral calculus mentioned in Section 2.

### 5.1 Finite Operator Description: a Realization

As was mentioned in Subsection 2.1 the main ingredients of the approach are polynomial sequence of binomial type and shift invariant operators. As was already pointed in Examples 3.15, 3.21 and 5.3 the notion of token is a useful refinement of polynomial sequence of binomial type. It is much more useful if we consider it together with shift invariant operators on c-semigroups not just groups. Next Subsections contains some details of this.

### 5.2 Hopf Algebra Description: a Realization

We describe some applications of c-semigroups and tokens. First we can add new lines to Proposition 3.11:

**Proposition 5.1** *Let  $C_2$  will be a c-semigroup with a right source  $e$ . Let also  $C_1$  be another c-semigroup and  $t(c_1, c_2)$  be a token between them. Then the list in the Proposition 3.11 for  $C_2$  can be extended via  $t$ -transform by the*

4. *The subspace of convolutions over  $C_1$ .*
5. *The subspace of linear functional over  $\mathcal{C}(C_1)$ .*

PROOF. By Theorem 3.26 the space of convolutions over  $C_2$  can be mapped to the space of convolutions over  $C_1$  via  $t$ -transform. A backward mapping from convolutions over  $C_1$  to convolutions over  $C_2$  is given by the dual token  $q(c_2, c_1)$  of associated delta family  $Q(c_1)$ .  $\square$

We establish the correspondence of four mentioned sets as linear spaces. However two of them (convolutions over  $C_1$  and  $C_2$ ) have the isomorphic structures as convolution algebras. This allows us to transfer also their multiplication law on the space of linear functionals.

**Definition 5.2** The product  $l = l_1 * l_2$  of two functionals  $l_1$  and  $l_2$  is again a linear functional corresponding to the convolution  $S = S_1 * S_2$ , where convolutions  $S_1$  and  $S_2$  correspond to  $l_1$  and  $l_2$ . It acts on a function  $p(c_1)$  as follows:

$$\langle l, p(c'_1, \cdot) \rangle = \int_{C_1} \langle l_1, p(c_1, \cdot) \rangle \langle l_2, p([c_1^{-1}c'_1], \cdot) \rangle dc_1$$

**Example 5.3** We return to the notations of Examples 3.15 and 3.21. For a linear functional  $l$  on  $C_2 = \mathbb{R}$  we introduce its t-transform  $[\mathcal{T}l](n)$  as its action (with respect to  $x$  variable) on the token  $t(n, x) = p_n(x)$  given by a sequence of polynomial type  $p_n(x)$ :

$$[\mathcal{T}l](n) = lT(n, x) = lp_n(x).$$

Then the product  $l$  of two functionals  $l_1$  and  $l_2$  are defined by the convolution on  $\mathbb{N}_+$ :

$$[\mathcal{T}l](n) = \int_{\mathbb{N}_+} l_1(k) l_2([k^{-1}n]) dn = \sum_{k=0}^n l_1(k) l_2(n - k). \quad (5.1)$$

The above definition of product can be found in [16]. The c-semigroup algebra over  $\mathbb{N}_+$  can be naturally realized as the multiplicative algebra of a formal power series of one variable  $t$  (see Proposition 3.6). So we obtain one more face an umbral algebra as follows:

**Corollary 5.4** [16] *The algebra of linear functionals on  $\mathbb{R}$  with the Hopf multiplication (5.1) is in a natural correspondence with the algebra of formal power series in one variable  $t$*

$$\{l_n\}_{n=0}^{\infty} \mapsto \sum_{n=0}^{\infty} l_n t^n.$$

### 5.3 The Semantic Description: a Realization

We start from the c-semigroup  $S$ , which is the direct product of many copies of the c-semigroup  $\mathbb{N}_+$ . The different copies of  $\mathbb{N}_+$  are labelled by letters of

an alphabet  $A$ , which is denoted by Greek letters. An element  $k$  of a copy  $\mathbb{N}_+$  labelled by a letter  $\alpha$  can be written interchangeably as  $k_\alpha$  or  $\alpha^k$ . In this sense any function  $p(k_\alpha, l_\beta, \dots)$  in  $\mathcal{C}_0(S)$  can be identified with the polynomial

$$\mathbf{p}(\alpha, \beta, \dots) = \sum_{k_\alpha, l_\beta, \dots} p(k_\alpha, l_\beta, \dots) \alpha^k \beta^l \dots$$

More over this identification send convolution  $p * q$  of two functions  $p, q \in \mathcal{C}_0(S)$  over  $S$  to the product of polynomials  $\mathbf{p}q$ . So we will not distinguish a function  $p(k_\alpha, l_\beta, \dots)$  and the corresponding polynomial  $\mathbf{p}(\alpha, \beta, \dots)$  any more as well as the convolution algebra  $\mathcal{C}_0(S)$  and polynomial ring  $D[A]$  over alphabet  $A$ .

A linear functional eval is defined as

$$\text{eval}(p) = \int_S p(k_\alpha, l_\beta, \dots) ds$$

via a measure  $ds$  on  $S$  such that  $ds(1_\alpha) = 1$  for any  $\alpha \in A$ . Because every  $p$  has a compact (i.e., finite) support in  $S$  the integration does not generate any difficulties. The equality (2.4) express the fact that  $S$  is the direct sum of  $\mathbb{N}_+$ 's and the measure  $ds$  is the direct product of corresponding measures  $ds_\alpha$ .

In spirit of 2.1.4 we require that the measure  $ds_\epsilon$  on the copy  $\mathbb{N}_+$  labelled by the distinguished umbra  $\epsilon$  is defined by the sequence  $(1, 0, 0, 0, \dots)$ .

Now eval defines an equivalence relation—umbral equivalence on  $\mathcal{C}_0(S)$ , namely  $f \simeq g$ ,  $f, g \in \mathcal{C}_0(S)$  if  $(f - g)$  belongs to the kernel of eval (i.e.,  $\text{eval}(f - g) = 0$ ). Because eval is additive but not multiplicative we see that umbral equivalence is invariant under addition but multiplication (unlike the support of functions belongs to different components in the direct product of  $\mathbb{N}_+$ ).

Finally we will define  $n.\beta$  as a function

$$\beta^1 \otimes \beta^0 \otimes \dots \otimes \beta^0 + \beta^0 \otimes \beta^1 \otimes \dots \otimes \beta^0 + \beta^0 \otimes \beta^0 \otimes \dots \otimes \beta^1$$

in the  $n$ -th tensor power  $\otimes_{k=1}^n \mathbb{N}_+(\beta)$  of a copy of  $\mathbb{N}_+$  labeled by  $\beta$  (this means that eval has the identical distribution on all of them).

This gives a realization for the classical umbral calculus described via semantic approach in [18, 19].

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